Multi-stage Rocket Fundamentals

Basic Staging Concept

Staging Benefits

Staging General Formulation

Staging General Solution Steps

Need for Optimality & Optimality Formulation

Unconstrained Optimization
Multi-staging in rockets is the design strategy that aims to make the ascent mission more efficient.

This is done by reducing the inert mass at regular intervals so that the overall efficiency of energy conversion increases significantly.

Multi-staging directly reduces the energy loss due to gravity, by shedding the inert mass.

It also permits use of dissimilar launch technologies within the same vehicle mission. However, multi-staging makes the mission design and implementation, more complex.
Consider a **rocket having** following parameters. $m_0 = 79.4$ Tons, $m_p = 60$ Tons, $I_{sp} = 241$ s, $g_0 = 9.81 \text{m/s}^2$, Payload mass = 9.4 Tons, $\beta_0 = 600 \text{ kg/s}$. Assume equal stages. Generate **both ideal & constant gravity** solutions.

\[
\begin{align*}
V_{0-1\text{stage}} &= 3331.7 \text{m/s}; \\
V_{0-2\text{stage}} &= 3783.9 \text{m/s} \\
V_{0-3\text{stage}} &= 3952.4 \text{m/s}; \\
V_{0-\infty\text{stage}} &= ?
\end{align*}
\]

\[
\begin{align*}
V_b (1-\text{Stage}) &= 2350.0 \text{ m/s}; \\
V_b (2-\text{Stage}) &= 2803.0 \text{ m/s}; \\
V_b (3-\text{Stage}) &= 2979.4 \text{ m/s}; \\
h_b (1-\text{Stage}) &= 79.7 \text{ km} \\
h_b (2-\text{Stage}) &= 54.3 \text{ km} \\
h_b (3-\text{Stage}) &= 45.1 \text{ km}
\end{align*}
\]

How does one **arrive at stage configurations?**
Basic Features of Staging

With increasing number of stages, though overall velocity increases, stage-wise increments keep reducing.

We see that for a given velocity, payload fraction also increases more slowly for higher No. of stages.
Basic Features of Staging

It is also found that for a given No. of stages, the effect of reducing inert mass \( r_s \) is more pronounced.

Thus, it is useful to adopt technologies that reduce \( r_s \).
Staging Philosophy

Staging is carried out under the following guidelines.

Each stage is a rocket by itself, with its own propellant and structure (including systems) mass.

Propellant type in each stage can be different.

The final payload mass is separately prescribed.

For $i^{th}$ stage operation of an N-stage rocket, the sum of masses of all the stages from $i+1$ to $n$ and the final payload mass is treated as its payload.

Inert mass of $i^{th}$ stage is separated after its burnout but before the operation of the $i + 1^{th}$ stage is begun.
Consider the following rocket configuration.

**Multi-stage Formulation**

- **Lift-off Mass:** \( m_0 = \sum_{i=1}^{n} m_{pi} + \sum_{j=1}^{n} m_{si} + m_* \)
- **Stage-wise Starting Mass:** \( m_{0i} = m_{pi} + m_{si} + m_{0i+1} \)
- **Stage-wise Ending Mass:** \( m_{fi} = m_{si} + m_{0i+1} \)
- **Stage-wise Specific Impulse:** \( I_{spi} \)
Multi-stage Formulation

Ideal burnout velocity can be obtained as follows.

Payload Fraction: \( \pi_* = \frac{m_*}{m_0} = \frac{m_*}{m_{0n}} \times \frac{m_{0n}}{m_{0(n-1)}} \times \cdots \times \frac{m_{02}}{m_{01}} \)

\[
= \prod_{i=1}^{n} \pi_i; \quad \pi_i = \frac{m_{0i+1}}{m_{0i}} \rightarrow \text{Stage Payload Ratio}
\]

Stage-wise Structural Ratio: \( \epsilon_i = \frac{m_{si}}{m_{si} + m_{pi}} \)

Ideal Burnout Velocity: \( V_b = \sum_{i=1}^{n} \Delta V_i = -\sum_{i=1}^{n} g_0 I_{spi} \ln \frac{m_{0i} - m_{pi}}{m_{0i}} \)

The ideal burnout velocity is in terms of the payload mass and structural ratios, apart from specific impulses.
Multi-stage Formulation

Ideal burnout velocity can be rewritten in terms of stage structural and payload ratios, as follows.

\[
\frac{m_{0i} - m_{pi}}{m_{oi}} = \frac{m_{si} + m_{0i+1}}{m_{oi}} = \frac{m_{0i+1}}{m_{oi}} + \frac{m_{si}}{m_{oi}}
\]

\[
= \pi_i + \left( \frac{m_{si}}{m_{si} + m_{pi}} \right) \times \left( \frac{m_{0i} - m_{0i+1}}{m_{0i}} \right)
\]

\[
= \pi_i + \varepsilon_i \times (1 - \pi_i) = \varepsilon_i + \pi_i \times (1 - \varepsilon_i)
\]

\[
V_b (Ideal) = -g_0 \sum_{i=0}^{n} I_{spi} \ln \left( \pi_i + \varepsilon_i \times (1 - \pi_i) \right)
\]

\[
= -g_0 \sum_{i=0}^{n} I_{spi} \ln \left( \varepsilon_i + \pi_i \times (1 - \varepsilon_i) \right)
\]
Staging problem definition starts with the mission specifications in terms of the payload mass \( (m_*) \), injection velocity \( (V_b) \), inclination \( (\theta_b) \) and altitude \( (h_b) \).

In addition, structural / system technologies are used to specify the structural (non-propellant) mass ratios \( \varepsilon_i \).

Further, propulsion technologies are used to create a selection set of \( I_{spi} \).

Lastly, a broad design objective in terms of the No. of stages is prescribed.

Design process provides the solutions for the total lift-off mass \( (m_0) \) and the stage mass configuration \( (m_{si} & m_{pi}) \).
Multi-stage Solution Steps

It is possible to use **staging specifications** $m_\ast$ and $\varepsilon_i$, along with **solution for**, $\pi_i$, to determine stage details.

\[
\varepsilon_i = \frac{m_{si}}{m_{si} + m_{pi}}; \quad m_{si} + m_{pi} = \frac{1}{\varepsilon_i} m_{si} \\
1 = \frac{m_{si} + m_{pi} + m_{0i+1}}{\pi_i} m_{0i+1}; \quad m_{si} + m_{pi} = \left(1 - \frac{\pi_i}{\pi_i}\right) m_{0i+1}
\]

These are **recursive relations** solved as shown below.

\[
m_{si} = \varepsilon_i m_{0i+1} \left(1 - \frac{\pi_i}{\pi_i}\right); \quad m_{pi} = (1 - \varepsilon_i) m_{0i+1} \left(1 - \frac{\pi_i}{\pi_i}\right); \\
m_0 = m_\ast + \sum_{i=0}^{n} m_{0i+1} \left(1 - \frac{\pi_i}{\pi_i}\right)
\]
Let us consider a **2-stage case** to understand the method.

\[
\begin{aligned}
m_{s2} + m_{p2} &= \frac{m_{s2}}{\mathcal{E}_2}; & \quad \frac{1}{\pi_2} &= \frac{m_{s2} + m_{p2} + m^*}{m^*} = 1 + \frac{m_{s2} + m_{p2}}{m^*} \\
m_{s2} &= \mathcal{E}_2 \left(\frac{1 - \pi_2}{\pi_2}\right) m^*; & \quad m_{p2} &= (1 - \mathcal{E}_2) \left(\frac{1 - \pi_2}{\pi_2}\right) m^*
\end{aligned}
\]

\[
\begin{aligned}
m_{02} &= \mathcal{E}_2 m^* \left(\frac{1 - \pi_2}{\pi_2}\right) + (1 - \mathcal{E}_2) m^* \left(\frac{1 - \pi_2}{\pi_2}\right) + m^* \\
m_{s1} &= \mathcal{E}_1 m_{02} \left(\frac{1 - \pi_1}{\pi_1}\right); & \quad m_{p1} &= (1 - \mathcal{E}_1) m_{02} \left(\frac{1 - \pi_1}{\pi_1}\right) \\
m_{01} &= m_{02} + \mathcal{E}_1 m_{02} \left(\frac{1 - \pi_1}{\pi_1}\right) + (1 - \mathcal{E}_1) m_{02} \left(\frac{1 - \pi_1}{\pi_1}\right) = m_0
\end{aligned}
\]
Two-stage Solution Example

A rocket has following parameters. $m_* = 0.4$ Tons, $\varepsilon_1 = \varepsilon_2 = 0.143$, $\pi_1 = 0.118$, $\pi_2 = 0.044$. Determine $m_0$.

\[
m_{s2} = \varepsilon_2 \left( \frac{1 - \pi_2}{\pi_2} \right) m_* = 0.143 \times 0.4 \times 21.727 = 1.243;
\]

\[
m_{p2} = (1 - \varepsilon_2) \left( \frac{1 - \pi_2}{\pi_2} \right) m_* = 0.857 \times 0.4 \times 21.727 = 7.448;
\]

\[
m_{02} = m_{p2} + m_{s2} + m_* = 9.091;
\]

\[
m_{s1} = \varepsilon_1 m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 0.143 \times 9.091 \times 7.474 = 9.716;
\]

\[
m_{p1} = (1 - \varepsilon_1) m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 0.857 \times 9.091 \times 7.474 = 58.230;
\]

\[
m_0 = m_{01} = m_{02} + m_{p1} + m_{s1} = 77.037
\]
Falcon has following parameters. \( m_\ast = 0.67 \, \text{T}, \, \varepsilon_1 = 0.058, \, \varepsilon_2 = 0.096, \, \pi_1 = 0.165, \, \pi_2 = 0.178 \). Determine \( m_0 \).

\[
m_{s_2} = \varepsilon_2 \left( \frac{1 - \pi_2}{\pi_2} \right) m_\ast = 0.096 \times 0.668 \times 4.714 = 0.302;
\]

\[
m_{p_2} = (1 - \varepsilon_2) \left( \frac{1 - \pi_2}{\pi_2} \right) m_\ast = 0.904 \times 0.668 \times 4.714 = 2.847;
\]

\[
m_{02} = m_{p_2} + m_{s_2} + m_\ast = 3.817;
\]

\[
m_{s_1} = \varepsilon_1 m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 0.058 \times 3.817 \times 5.061 = 1.12;
\]

\[
m_{p_1} = (1 - \varepsilon_1) m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 0.942 \times 3.817 \times 5.061 = 18.197;
\]

\[
m_0 = m_{01} = m_{02} + m_{p_1} + m_{s_1} = 23.13
\]
Cyclone has following parameters. $m_*= 4.1$ T, $\varepsilon_1 = 0.049$, $\varepsilon_2 = 0.067$, $\varepsilon_3 = 0.304$, $\pi_1 = 0.324$, $\pi_2 = 0.142$, $\pi_3 = 0.471$.

\[
m_{s3} = \varepsilon_3 \left( \frac{1 - \pi_3}{\pi_3} \right) m_* = 1.4; \quad m_{p3} = (1 - \varepsilon_2) \left( \frac{1 - \pi_2}{\pi_2} \right) m_* = 3.204;
\]

\[
m_{03} = m_{p3} + m_{s3} + m_* = 8.705; \quad m_{s2} = \varepsilon_2 \left( \frac{1 - \pi_2}{\pi_2} \right) m_{03} = 3.524;
\]

\[
m_{p2} = (1 - \varepsilon_2) \left( \frac{1 - \pi_2}{\pi_2} \right) m_{03} = 49.07; \quad m_{02} = m_{p1} + m_{s1} + m_{03} = 61.3;
\]

\[
m_{s1} = \varepsilon_1 m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 6.265; \quad m_{p1} = (1 - \varepsilon_1) m_{02} \left( \frac{1 - \pi_1}{\pi_1} \right) = 121.6;
\]

\[
m_0 = m_{01} = m_{02} + m_{p1} + m_{s1} = 189.2
\]
Payload ratios, as was seen earlier, are part of the design solution and hence need to be determined through a consistent procedure.

Different solutions for payload ratios result in different performances and among the many such performances, there are a few that give the best performance.

This has given rise to an optimization based methodology for arriving at the values of payload ratios.
**Basis for Optimality**

In multi-stage rocket design, basic design variables are No. of stages and mass distribution within each stage.

Payload mass and burnout velocity can either be objective function or constraint.

In most design problems, objective is to maximize, either the payload mass for a given burnout velocity or maximize burnout velocity for a given payload mass.

In some cases, multi-stage design criteria could be to minimize either total lift-off mass or total propellant, for the specified payload mass and burnout velocity. In most cases, No. of stages is used as a parameter.
In all such design exercises, the technology, in terms of the structural configuration, materials, propellant etc., is treated as specified constants.

Also, as burnout altitude / inclination are in some way linked to ideal burnout velocity, optimization strategy based only on ideal burnout velocity is adequate.

\[
V_b (Ideal) = -g_0 \sum_{i=1}^{n} I_{spi} \ln \left( \pi_i + \varepsilon_i \times (1 - \pi_i) \right)
\]

\[
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\]

It is seen that \( V_b \) is a monotonic function of \( I_{spi} \) and \( \varepsilon_i \).
If $V_b$ (or $m_*$) is taken as the objective function, it is possible to set up the optimization problem as ‘Maximize the $V_b$, while keeping $m_*$ constant’.

$$V_b(\pi_i) = \ln \left\{ \varepsilon_i + \pi_i (1 - \varepsilon_i) \right\}$$

$$dV_b \over d\pi_i = -g_0 I_{spi} \frac{(1 - \varepsilon_i)}{(\varepsilon_i + \pi_i \times (1 - \varepsilon_i))} = 0$$

$$\varepsilon_i = 1 \text{ (No Propellant?)}; \quad V_b = 0$$

**What is wrong** in the above way of solving?
Consider an alternate way of optimizing the multi-stage configuration.

\[
m_\star (\pi_i) = m_0 \pi_\star; \quad m_0 = m_\star + \sum_{i=1}^{n} m_{0i+1} \left( \frac{1-\pi_i}{\pi_i} \right); \\
\frac{dm_\star}{d\pi_i} = \frac{m_\star}{m_0} \frac{dm_0}{d\pi_i} + \frac{m_\star}{\pi_i} = 0; \quad \pi_i = \infty \ (\text{Zero } m_0)?
\]

It is an inconsistent solution as ‘\(\pi_i\)’ cannot be > 1.

This is so because, problem becomes consistent only if ‘\(\pi_i\)’s satisfy the ‘\(\pi_\star\)’ constraint.
Summary

Multi-staging is an important concept for improving the efficiency of an ascent mission, which is achieved by reducing the inert mass at each stage of operation.

Multi-staging, however, increases the complexity of the assembly and operation of the rocket.

Multistage solution involves obtaining the stage payload ratios, for given burnout performance and the structural ratios, while also providing the total lift-off mass.

Optimization is a convenient way of arriving at multi-stage configuration, while achieving mission objectives.