Planar Time-Optimal and Length-Optimal Paths under Acceleration Constraints

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ABSTRACT

This paper considers the problems of finding minimum time and minimum length trajectories for a vehicle moving in a two-dimensional plane from a given initial position and velocity to a specified terminal condition under a magnitude constraint on the acceleration. Unlike previous work on related problems, variations in the magnitude as well as direction of the velocity vector are considered. The time-optimal problem is solved for three different terminal conditions: 1) fixed terminal heading with the terminal position being free; 2) fixed terminal position with the terminal velocity being free and 3) a terminal condition involving both position and velocity. The Pontryagin’s maximum principle is used to show that, for each kind of terminal condition, the time-optimal paths are either straight lines or parabolic arcs. The length-optimal problem is solved for a fixed terminal heading with the terminal position being free. The Pontryagin’s maximum principle is used to show that the corresponding length optimal paths possess a special property whereby the angle bisector between the acceleration and velocity vectors is a constant. This property is used to obtain a feedback strategy for generating the optimal acceleration vector. A numerical example is used to compare the time-optimal and length-optimal paths for the same terminal conditions.

Keywords: Time-optimal, Length-optimal, Pontryagin maximum principle.

1 INTRODUCTION

Path planning for autonomous agents, be it a ground vehicle, air vehicle or a robot, is a popular area of research. Planning the motion of such autonomous vehicles is very critical for the full utilization of their potential, an example of which could be the situation where the vehicle is required to maneuver a lot with a limited quantity of fuel. Path planning arises in a number of aerospace applications. For example, the automatic guidance of an aircraft in a horizontal plane as described by Erzberger and Lee\(^1\), usually involves tasks of three types: steering the aircraft from any initial position and velocity to a) any terminal position with a specified terminal heading; b) a specified terminal position with arbitrary terminal heading; and c) intercept and fly along a specified direction. Problems of these types occur when the air vehicle has to capture an ILS beam at a specified point or when it has to reach and fly along a particular VOR radial. In such applications, it is of interest to find a minimum time path or a path of minimum length between the initial and terminal positions. Another example, given in the paper by Yang and Kapila\(^2\), is the cooperative mission planning of multiple Unmanned Air Vehicles (UAV)
operating in a multi target environment which necessitates computation of optimal path plans for a vehicle-target pair.

The seminal paper on optimal path planning was published by Dubins \(^3\). Dubins showed that curves of minimal length with a constraint on average curvature between specified positions and tangents can be constructed using combinations of straight line segments and circular arcs. Moreover, such curves consist of only three segments where the first and third segments are circular arcs of radius equal to the minimum turn radius and the second segment is either an arc of a circle of minimum turn radius or a line segment. Dubins’ work was later interpreted by Bui, \textit{et al} \(^4\) as the problem of finding the shortest continuously differentiable path taken by a vehicle moving in a two-dimensional plane from a given initial position and heading to a specified terminal position and heading at constant speed under a constraint on the turn radius. The results of Dubins’ paper were later independently proved by Boissonnat, \textit{et al} \(^5\) by applying the Pontryagin’s maximum principle \(^{12–15}\). Modgalya and Bhat \(^6\) considered the Dubins’ problem and gave a feedback guidance strategy which uses only the instantaneous position and heading of the vehicle to calculate the instantaneous turn rate command required to calculate the time-optimal path to the destination point.

Reeds and Shepp \(^7\) extended Dubins’ work by finding the shortest paths for a vehicle that moves in a plane with constant speed and can instantaneously switch between forward and backward directions. In Dubins’ case the shortest paths consist of combinations of circular arcs of minimum turn radius and line segments. However, unlike Dubins’ case the candidate optimal paths consist of at most five segments where the middle segment is a straight line and the other segments are circular arcs of minimum turn radius with cusps between successive segments being allowed.

The time-optimal paths for the vehicle described by Dubins \(^3\) and Reeds and Shepp \(^7\) have a discontinuous curvature profile as they are essentially concatenations of circular arcs and line segments. Anisi, \textit{et al} \(^8\) applied the Pontryagin Maximum Principle to generate nearly time-optimal but continuous curvature paths by minimizing a cost function that penalises rapid variations in the acceleration.

Erzberger and Lee \(^1\) considered the automatic guidance of an aircraft in a cruise flight, where the forces acting on it are balanced and it is moving with a constant velocity in a horizontal plane with an upper limit on its turning radius. The paper considers two kinds of guidance problems in addition to the one considered by Dubins, in which the aircraft has to be steered from its initial position and heading to either intercept or fly along a line or to a final position with arbitrary heading. The Pontryagin’s maximum principle was used to reduce the optimal guidance problem to a two-point boundary value problem and to geometrically characterize time-optimal paths for each of the three guidance problems. It was found that the time-optimal trajectories consist of connected segments of straight lines and circular arcs of minimum turn radius.

Yang and Kapila \(^2\) considered the optimal path-planning problem of finding the shortest paths for a UAV moving in a two-dimensional plane from a given initial position and heading to a specified terminal position and heading at constant speed under a constraint on the turn radius and additional tactical constraints arising in target touring and obstacle avoidance where the path of the UAV was restricted to maintain a certain minimum distance from threats like
enemy radar or obstacles arising due to the geography of the place like mountains, civilian centers etc. The results of Dubins’ paper were used to reduce the path planning problem to a parameter optimization problem. The optimal paths are combinations of straight lines and circular arcs of minimum turn radius but contain more than three segments unlike Dubins’ case due to restrictions imposed in the path of the vehicle. A necessary condition for optimality in the presence of tactical constraints was characterized and an efficient numerical algorithm was developed to enforce the optimality criteria.

The references mentioned above assumed that the velocity of the vehicle is fixed in magnitude. Shapira and Ben-Asher\textsuperscript{9} extended the class of path planning problems by replacing the constant velocity assumption with a velocity model involving an instantaneously changing velocity which is assumed to be a linear function of the turn rate magnitude. The dependence of velocity on the turn rate implies that the velocity as well as the turn rate are piecewise constant rather than continuous functions of time.

In this paper we consider the path planning problem of optimally steering a vehicle moving in a two-dimensional plane from a given initial point and velocity to a specified terminal condition on position and/or heading subject to a magnitude constraint on acceleration. As a departure from the previous work\textsuperscript{1–8} we do not require the speed of the vehicle to be constant. While the constant speed case is pertinent to aircraft, our problem is relevant to a terrestrial vehicle moving on a smooth floor that provides limited friction or to a spacecraft moving in free space under the action of a gimballed thruster of limited capacity. In both the examples, the limited acceleration capability can be partially used to brake, accelerate, or turn, and the optimal combination of braking, accelerating or turning is not obvious. Moreover, unlike in the case of motion at constant speed, shortest paths are different from quickest paths when speed variation is allowed. Hence we consider time-optimal as well as length-optimal problems.

In Section 2, we consider the trajectory of a vehicle moving in a two-dimensional plane from any initial point and velocity to a specified terminal condition in the minimum possible time under a magnitude constraint on acceleration. We consider three different types of terminal conditions; a) terminal heading fixed, terminal position free; b) terminal position fixed, terminal heading free; c) mixed constraint involving both terminal position and terminal heading. We use the Pontryagin’s maximum principle to show that, for each of the three terminal conditions, the time-optimal acceleration profile is a constant. Consequently, time-optimal paths are either straight lines or parabolic arcs. A constant acceleration profile is a special case of the Bilinear Tangent Law\textsuperscript{11} which arises in the problem of minimizing a cost function involving the final time and state for a particle moving in a two-dimensional plane under the action of a constant thrust at a variable angle of attack. By applying the Pontryagin’s maximum principle, the tangent of the optimal thrust angle is obtained as a ratio of two linear functions of time. In the case where time is to be minimized, the expression for the thrust angle reduces to a constant and thus the optimal acceleration profile takes a constant value.

In Section 3, we consider the trajectory of a vehicle moving in a two-dimensional plane from any initial point and velocity to a specified final heading with its final position being free such that the arc length of the path traversed by the vehicle is a minimum under a magnitude constraint on the acceleration. We apply Pontryagin’s maximum principle to show that the optimal trajectories possess a special property; the angle bisector between the acceleration and
velocity vectors along an optimal trajectory is fixed in direction. This property directly yields a
feedback synthesis for the optimal acceleration profile.

In Section 4, we use a numerical example to compare the minimum time paths with the
minimum arc length paths for the case of free terminal position and fixed terminal heading. We
compare the plots of the optimal paths in the plane, magnitude of velocity versus time, the time
taken to achieve a specified terminal heading and corresponding distance covered as functions
of the heading angle for the two problems. As expected, the solution to the length optimal
problem involves moving at lower speeds.

2 TIME-OPTIMAL PATHS

Consider a vehicle moving in a two-dimensional plane. The kinematic equations of the
vehicle are

\[ \begin{align*}
\dot{x} &= v, \\
\dot{v} &= a,
\end{align*} \tag{2.1} \tag{2.2} \]

where \( x \in \mathbb{R}^2 \) is the vector of instantaneous coordinates of the vehicle with respect to a suitable
pair of orthogonal axes, \( v \in \mathbb{R}^2 \) is the vector of its instantaneous velocity components and \( a \in \mathbb{R}^2 \)
the vector of acceleration components. We wish to determine the time history for the
acceleration \( a \) that minimizes the cost function,

\[ C(a) = \int_0^{t_f} d\tau, \tag{2.3} \]

subject to the magnitude constraint \( \|a(t)\| \leq M \) for all \( t \in [0,t_f] \) and the constraint that the
solutions of (2.1)-(2.2) satisfy the specified initial conditions \( x(0) = x_i, v(0) = v_i \) and specified
inequality and equality terminal state constraints of the form

\[ \begin{align*}
g(x(t_f), v(t_f)) &\geq 0, \\
h(x(t_f), v(t_f)) &= 0,
\end{align*} \tag{2.4} \tag{2.5} \]

where \( g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) and \( h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^n \) are smooth functions, for some \( n \).

In order to write down the necessary conditions for time optimality, we introduce the
Hamiltonian function for the equations (2.1)-(2.2) given by

\[ H(x, v, \lambda_x, \lambda_v, a) = -1 + \lambda_x^T v + \lambda_v^T a, \tag{2.6} \]

where \( \lambda_x \in \mathbb{R}^2 \) and \( \lambda_v \in \mathbb{R}^2 \) are the adjoint vectors corresponding to \( x \) and \( v \), respectively. If \( a^* \)
is the time-optimal acceleration history and \( x^*, v^* \) represent the solutions of (2.1) and (2.2) for
\( a = a^* \) then, according to Pontryagin’s maximum principle, \( a^* \) satisfies the equations

\[ \begin{align*}
H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) &= 0, \\
H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) &\geq H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a),
\end{align*} \tag{2.7} \tag{2.8} \]
for every $t \in [0, t_f]$ and every $a \in \mathbb{R}^2$ satisfying $\|a\| \leq M$, where the adjoint time histories $\lambda_\alpha^*, \lambda_v^*$ satisfy the equations

$$\dot{\lambda}_\alpha^*(t) = -\frac{\partial H}{\partial x}(x^*(t), v^*(t), \lambda_\alpha^*(t), \lambda_v^*(t), a^*(t)) = 0, \quad (2.9)$$

$$\dot{\lambda}_v^*(t) = -\frac{\partial H}{\partial v}(x^*(t), v^*(t), \lambda_\alpha^*(t), \lambda_v^*(t), a^*(t)) = -\lambda_\alpha^*(t), \quad (2.10)$$

and the transversality conditions

$$\lambda_\alpha^*(t_f) = \frac{\partial g}{\partial x}(x^*(t_f), v^*(t_f))\alpha + \frac{\partial h}{\partial x}(x^*(t_f), v^*(t_f))\beta, \quad (2.11)$$

$$\lambda_v^*(t_f) = \frac{\partial g}{\partial v}(x^*(t_f), v^*(t_f))\alpha + \frac{\partial h}{\partial v}(x^*(t_f), v^*(t_f))\beta, \quad (2.12)$$

where $\beta \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^+$ satisfies $\alpha g(x(t_f), v(t_f)) = 0$, with $\frac{\partial h}{\partial x} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2 \times n}$ and $\frac{\partial h}{\partial v} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2 \times n}$ denoting the Jacobian matrices of $h$ with respect to $x$ and $v$, respectively. Equations (2.6) and (2.8) together imply that

$$a^*(t) = M\|\lambda_v^*(t)\|^{-1}\lambda_v^*(t). \quad (2.13)$$

We now construct time optimal paths for the vehicle by applying the necessary conditions (2.7)-(2.13) for various terminal constraints of the form (2.4)-(2.5).

### 2.1 Fixed terminal heading, Free terminal position

First we consider the problem of steering the velocity vector in equation (2.2) to a specified terminal heading. If $e_1 \in \mathbb{R}^2$ is a unit vector along the desired terminal heading and $e_2 \in \mathbb{R}^2$ satisfies $e_1^Te_2 = 0$, then the terminal conditions are given by (2.4) and (2.5) where $n = 1$ and

$$g(x, v) = e_1^Tv, \quad (2.14)$$

$$h(x, v) = e_2^Tv. \quad (2.15)$$

Upon substituting (2.14) and (2.15), the transversality conditions (2.11) - (2.12) yield

$$\lambda_\alpha^*(t_f) = 0, \quad (2.16)$$

$$\lambda_v^*(t_f) = \alpha e_1 + \beta e_2, \quad (2.17)$$

$$\alpha e_1^Tv^*(t_f) = 0. \quad (2.18)$$

From (2.9), (2.10) and (2.16), it can be deduced that $\lambda_\alpha^*(t)$ is constant for all $t \in [0, t_f]$ and hence the optimal acceleration profile $a^*(t)$ is also constant. Note that the terminal conditions (2.4) and (2.5) under (2.14)-(2.15) are satisfied with equality if the terminal velocity of the vehicle is zero and with strict inequality if the terminal velocity has a positive component along $e_1$.

First consider the case when the optimal trajectory terminates in rest. Since the terminal velocity is zero, (2.18) is satisfied for every $\alpha \in \mathbb{R}^+$. Therefore, there exist $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that the constant adjoint vector is given by (2.17). The optimal acceleration time history thus takes a constant value $a^*(t) \equiv a^*$ satisfying $\|a^*\| = M$ and $a^*Te_1 \geq 0$. Since the only way to
bring the vehicle to rest using a constant acceleration vector is by letting the acceleration vector be opposite to the initial velocity, it follows that the optimal acceleration vector is given by,

$$a^* = -M||v_i||^{-1}v_i.$$  \hspace{1cm} (2.19)

The condition $a^*T_e1 \geq 0$ implies $v_iT_e1 \leq 0$. We therefore conclude that the optimal trajectory terminates in rest only if $v_iT_e1 \leq 0$, that is, if the heading change required is obtuse. In this case, the optimum acceleration time history is constant and opposite to the initial velocity vector, and the resulting optimal paths are straight lines.

Next, consider the case where the optimal trajectory does not terminate in rest. Equations (2.4), (2.5) and (2.18) are then satisfied only if $\alpha = 0$. This reduces (2.17) to $\lambda^*_e(t_f) = \beta e_2$. Thus the transversality condition (2.17) is satisfied only if the terminal adjoint vector $\lambda^*_e(t_f)$ is oriented along $e_2$, that is, the terminal adjoint vector is perpendicular to the final velocity. It follows from (2.13) that the acceleration vector is also perpendicular to the final velocity at the terminal time. The optimal acceleration time history thus takes a constant value $a^*(t) \equiv a^*$ satisfying $a^*T_e1 = 0$. This implies that the instantaneous velocity component along $e_1$ is a constant. It therefore follows that $v_iT_e1 = v^*_e(t_f)T_e1 > 0$. Thus it follows that the optimal trajectory does not terminate in rest only if the heading change required is acute.

Since $\frac{d}{dt}(a^*Tv^*(t)) = M^2 > 0$, it follows that the velocity component along the acceleration vector is increasing. However, this velocity component is zero at the terminal time. Hence $a^*Tv^*(t) < 0$ for all $t \in [0,t_f]$, that is, the angle between $a^*$ and $v^*(t)$ is obtuse for all $t \in [0,t_f]$. Thus from $v_iT_e1 > 0$, $a^*T_e1 = 0$ and $v_iT_a^* < 0$, it follows that $a^*$ is opposite to the component of $v_i$ perpendicular to $e_1$ and is given by

$$a^* = -M \left[ ||v_i||^2 - (v_iT_e1)^2 \right]^{-\frac{1}{2}} (v_i - (v_iT_e1)e_1).$$  \hspace{1cm} (2.20)

It is well known that the path traced by a point moving with a constant acceleration vector is a parabola whose axis is along the direction of the acceleration vector. The only point on the parabola where the axis is perpendicular to the tangent is the vertex of the parabola. Since $a^*Tv^*(t_f) = 0$, it follows that the optimal path is a parabolic arc that terminates at the vertex of the parabola.

The optimal strategy for the problem of steering the velocity vector to a specifiedterminal heading can thus be written as,

$$a^*(t) = -M||v_i||^{-1}v_i, \quad v_iT_e1 \leq 0,$n
n$$= -M \left[ ||v_i||^2 - (v_iT_e1)^2 \right]^{-\frac{1}{2}} (v_i - (v_iT_e1)e_1), \quad v_iT_e1 > 0. \hspace{1cm} (2.21)$$

The terminal time is the time required for the velocity component along the acceleration vector to vanish. Thus, the optimum time required to affect the required heading change is,

$$T(v_i) = M^{-1}||v_i||, \quad v_iT_e1 \leq 0,$n
n$$= -M^{-2} \left[ ||v_i||^2 - (v_iT_e1)^2 \right]^{\frac{1}{2}}. \quad v_iT_e1 > 0. \hspace{1cm} (2.22)$$

Figure (2.1) shows plots of trajectories generated by the time-optimal acceleration time history (2.21) for the initial conditions $x_i = [2,3]^T$ and $v_i = [10,0]^T$ for different terminal headings.
2.2 Fixed terminal position, Free terminal velocity

Next, we consider the problem of steering the position vector in (2.1) to a specified terminal position \(x_f\). The terminal conditions are given by (2.4) and (2.5) with \(n = 2\) and

\[
\begin{align*}
g(x,v) &= 0, \quad (2.23) \\
h(x,v) &= x - x_f. \quad (2.24)
\end{align*}
\]

Upon substituting equations (2.23) and (2.24), the transversality conditions (2.11) and (2.12) yield,

\[
\begin{align*}
\lambda_x^*(t_f) &= c, \quad (2.25) \\
\lambda_v^*(t_f) &= 0, \quad (2.26)
\end{align*}
\]

where \(c \in \mathbb{R}^2\) is a constant vector to be determined. Equations (2.9), (2.10), (2.25) and (2.26) yield

\[
\lambda_v^*(t) = c(t_f - t). \quad (2.27)
\]

Equation (2.13) now implies that the optimal acceleration profile is a constant given by

\[
a^*(t) \equiv a^* = M\|c\|^{-1}c. \quad (2.28)
\]

Since the path traced by a point moving with a constant acceleration is a parabola, it follows that the optimal paths are parabolic arcs described by the parametric equation

\[
x^*(t) = x_i + v_i t + 0.5a^* t^2. \quad (2.29)
\]

Letting \(J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\) and premultiplying (2.29) at \(t = t_f\) with \(a^*^T J\) and \(v_i^T J\), respectively, yields

\[
\begin{align*}
t_f &= \frac{a^*^T J(x_f - x_i)}{a^*^T v_i}, \quad (2.30) \\
(a^*^T J(x_f - x_i))^2 &= 2(v_i^T a^*)(v_i^T J(x_f - x_i)). \quad (2.31)
\end{align*}
\]
On letting $a^* = [M \cos \theta \quad M \sin \theta]^T$, (2.31) yields a quartic equation in $\tan^2 \theta$ which can be numerically solved to obtain up to four different values for the orientation $\theta$ of the acceleration vector. Among these values, the one that yields the minimum value for the terminal time (2.30) gives the orientation of the acceleration vector. Figure (2.2) shows optimal paths starting with the initial conditions $x_i = [0, 0]^T$, $v_i = [10, 0]^T$ for different destination points.

![Figure 2.2: Time-Optimal paths for various terminal positions](image)

### 2.3 Mixed constraint

Next we consider the problem of steering the solutions of (2.1) and (2.2) to a point $x_f$ with the final velocity $v_f$ in a minimum time $t_f$ such that at the terminal time, the velocity vector points directly towards a given point $x_a \in \mathbb{R}^2$. The terminal conditions are of the form (2.4) and (2.5) with $n = 1$ and

\[
\begin{align*}
g(x, v) &= (x_a - x)^T v, \quad \text{(2.32)} \\
h(x, v) &= (x_a - x)^T J v. \quad \text{(2.33)}
\end{align*}
\]

Substituting (2.32) and (2.33), the transversality conditions (2.11) and (2.12) yield

\[
\begin{align*}
\lambda_\alpha^*(t_f) &= -\alpha v^*(t_f) - \beta J v^*(t_f), \\
\lambda_\alpha^*(t_f) &= \alpha (x_a - x^*(t_f)) + \beta J (x_a - x^*(t_f)), \\
\alpha (x_a - x^*(t_f))^T v &= 0. \quad \text{(2.36)}
\end{align*}
\]

Note that the terminal conditions (2.4) and (2.5) under (2.32) and (2.33) are satisfied with equality if the vehicle comes to rest or $x_a = x(t_f)$ and with inequality if the vehicle has a non-zero speed at the terminal time and $x_a \neq x(t_f)$. We consider the case where the vehicle does not terminate in rest and $x_a \neq x(t_f)$. Equation (2.36) is then satisfied only if $\alpha = 0$. Upon substituting $\alpha = 0$ in (2.34) and (2.36), it can be seen that both $\lambda_\alpha^*(t_f)$ and $\lambda_\alpha^*(t_f)$ are perpendicular to $v^*(t_f)$. Moreover, we know from (2.9) and (2.10) that $\lambda_\alpha^*$ is a constant and hence

\[
\lambda_v^*(t) = -\lambda_\alpha^* f + \lambda_v^*(0) \quad \text{(2.37)}
\]
for all \( t \in [0, t_f] \). Premultiplying (2.37) by \( v^*(t_f) \) yields \( v^*(t_f)^T \lambda_t^*(t) = v^*(t_f)^T \lambda_t^*(0) \). This implies that \( v^*(t_f)^T \lambda_t^*(t) \) is a constant for all \( t \in [0, t_f] \). However, since \( v^*(t_f)^T \lambda_t^*(t) = 0 \), it follows that \( v^*(t_f)^T \lambda_t^*(t) = 0 \) for all \( t \in [0, t_f] \). Thus the instantaneous adjoint vector \( \lambda_t^*(t) \) is perpendicular to the final velocity vector and therefore, from (2.13), the optimal acceleration time history takes a constant value \( a^*(t) \equiv a^* \) satisfying \( \|a^*\| = M \) and \( a^T v^*(t_f) = 0 \). This implies that the instantaneous velocity component along \( v^*(t_f) \) is a constant which therefore leads to \( v_i^T v^*(t_f) = v^*(t_f)^T v^*(t_f) > 0 \). Thus it follows that using the optimal acceleration \( a^* \), the vehicle can point directly towards only those points \( x_a \) which satisfy at the terminal time \( v_i^T (x_a - x^*(t_f)) \geq 0 \).

Since the path traced by a point moving with a constant acceleration is a parabola, it follows that the optimal paths are parabolas and the instantaneous position and velocity of the vehicle are given by
\[
\begin{align*}
x^*(t) &= x_i + v_i t + 0.5 a^* t^2, \quad (2.38) \\
v^*(t) &= v_i + a^* t. \quad (2.39)
\end{align*}
\]
Premultiplying (2.38) and (2.39) at \( t = t_f \) with \( a^* T \) and using \( a^T (x^*(t_f) - x_a) = a^T v^*(t_f) = 0 \) gives
\[
\begin{align*}
a^T (x_i - x_a) + (a^T v_i) t_f + 0.5 a^* t_f^2 &= 0, \quad (2.40) \\
t_f &= -M^{-2} (a^T v_i). \quad (2.41)
\end{align*}
\]
Substituting (2.41) in (2.40) yields
\[
a^T (x_i - x_a) = 0.5 M^{-2} (a^T v_i)^2. \quad (2.42)
\]
Upon letting \( a^* = [M \cos \theta \quad M \sin \theta] \)T, (2.40) yields a quartic equation in \( \tan(\theta/2) \) which can be solved numerically to obtain up to four different values for the orientation \( \theta \) of the acceleration vector. Among these values, the one that yields the minimum value for the terminal time (2.41) is the orientation of the optimal acceleration vector. Figure (2.3) shows optimal paths with the initial conditions \( x_i = [0, 0]^T \), \( v_i = [10, 0]^T \) for different target points.

### 3 LENGTH-OPTIMAL PATHS

In this section, we consider the problem of determining the time history for the acceleration \( a \) of the vehicle that minimizes the cost function
\[
C(a) = \int_0^{t_f} \|v(\tau)\| d\tau \quad (3.1)
\]
subject to the magnitude constraint \( \|a(t)\| \leq M \) for all \( t \in [0, t_f] \) and the constraint that the solutions of (2.1)-(2.2) satisfy the specified initial conditions \( x(0) = x_i \), \( v(0) = v_i \) and inequality and equality terminal state constraints of the form (2.4) and (2.5) respectively where \( n = 1 \) and
\[
\begin{align*}
g(x, v) &= e_1^T v_i \quad (3.2) \\
h(x, v) &= e_2^T v_i \quad (3.3)
\end{align*}
\]
where \( e_1 \in \mathbb{R}^2 \) is a unit vector along the desired terminal heading and \( e_2 \in \mathbb{R}^2 \) satisfies \( e_1^T e_2 = 0 \).
The Hamiltonian function for the length-optimal problem is given by,

\[ H(x, v, \lambda_x, \lambda_v, a) = -\|v\| + \lambda_x^T v + \lambda_v^T a, \]  

(3.4)

where \( \lambda_x \in \mathbb{R}^2 \) and \( \lambda_v \in \mathbb{R}^2 \) are the adjoint vectors corresponding to the position \( x \in \mathbb{R}^2 \) and velocity \( v \in \mathbb{R}^2 \), respectively. If \( a^* \) is the optimal acceleration history and \( x^*, v^* \) represent the solutions of (2.1) and (2.2) for \( a = a^* \) then, according to Pontryagin’s maximum principle\(^{10} \), \( a^* \) satisfies the equations

\[ H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) = 0, \]  

(3.5)

\[ H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) \geq H(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a) \]  

(3.6)

for every \( t \in [0, t_f] \) and every \( a \in \mathbb{R}^2 \) satisfying \( \|a\| \leq M \), where the adjoint time histories \( \lambda_x^*(t), \lambda_v^*(t) \) satisfy

\[ \dot{\lambda}_x^*(t) = -\frac{\partial H}{\partial x}(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) = 0, \]  

(3.7)

\[ \dot{\lambda}_v^*(t) = -\frac{\partial H}{\partial v}(x^*(t), v^*(t), \lambda_x^*(t), \lambda_v^*(t), a^*(t)) = -\lambda_v^*(t) + \|v^*(t)\|^{-1}v^*(t), \]  

(3.8)

and the transversality conditions

\[ \lambda_x^*(t_f) = 0, \]  

(3.9)

\[ \lambda_v^*(t_f) = \alpha e_1 + \beta e_2, \]  

(3.10)

\[ \alpha e_1^T v^*(t_f) = 0, \]  

(3.11)

where \( \alpha \in \mathbb{R}^+ \) and \( \beta \in \mathbb{R} \). Equations (3.4) and (3.6) together imply that,

\[ a^*(t) = M\|\lambda_v^*(t)\|^2 \lambda_v^*(t). \]  

(3.12)
Equations (3.7) and (3.9) imply \( \lambda^*_v \equiv 0 \). Equation (3.8) now reduces to
\[
\dot{\lambda}^*_v(t) = \|v^*(t)\|^{-1}v^v(t), \ t \in [0, t_f)
\]
(3.13)

On substituting \( \lambda^*_v = 0 \) in (3.4), (3.5) yields \( \lambda^*_v(t)^Ta^*(t) = \|v^*(t)\| \) for every \( t \in [0, t_f) \). On further substituting (3.12), we get
\[
\|\lambda^*_v(t)\| = \frac{\|v^*(t)\|}{M}
\]
(3.14)

for every \( t \in [0, t_f) \). We next state a key property of the length optimal paths.

**Proposition 3.1 :** The angle bisector between the velocity vector and the acceleration vector is constant along every optimal trajectory.

**Proof:** Let \( b : [0, t_f) \rightarrow \mathbb{R}^2 \) be given by,
\[
b(t) = \|v^*(t)\|^{-1}v^*(t) + M^{-1}a^*(t).
\]
(3.15)

Equation (3.12) yields
\[
b(t) = \|v^*(t)\|^{-1}v^*(t) + \|\lambda^*_v(t)\|^{-1}\lambda^*_v(t).
\]
(3.16)

We compute
\[
\dot{b} = \|v^*(t)\|^{-1}v^v(t) - \|v^*(t)\|^{-3}(v^*(t)^Tv^*(t))v^*(t)
+ \|\lambda^*_v(t)\|^{-1}\lambda^*_v(t) - \|\lambda^*_v(t)\|^{-3}(\lambda^*_v(t)^T\lambda^*_v(t))\lambda^*_v(t).
\]
(3.17)

On using (3.12), (3.13), (3.14) and simplifying, (3.17) reduces to
\[
\dot{b} = \left[ \|\lambda^*_v(t)\|^{-1} - \|v^*(t)\|^{-1}\|\lambda^*_v(t)\|^{-2}v^*(t)^T\lambda^*_v(t) \right] b(t)
\]
(3.18)

Equation (3.18) implies that \( b(t) \) and \( \dot{b}(t) \) are parallel at every instant and therefore it follows that the unit vector \( \|b(t)\|^{-1}b(t) \) is constant. The result now follows by noting that the unit vector \( \|b(t)\|^{-1}b(t) \) is the unit vector along the angle bisector between the velocity vector and the acceleration vector along the optimal trajectory.

Proposition 3.1 implies that the instantaneous direction of the optimal acceleration vector can be obtained by reflecting the instantaneous direction of the velocity vector about the fixed direction of the angle bisector. Thus, if we let \( c = \|b(t_f)\|^{-1}b(t_f) \) denote the constant unit vector along the angle bisector, then
\[
M^{-1}a^*(t) = \|v^*(t)\|^{-1}Rv^*(t),
\]
(3.19)

where \( R = 2cc^T - I \in \mathbb{R}^{2 \times 2} \) is the orthogonal matrix representing reflection about the unit vector \( c \). We conclude that, for a vehicle moving in a two-dimensional plane with kinematics given by (2.1) and (2.2), the optimal acceleration vector for achieving the shortest path that yields a desired terminal heading is given by (3.19).

Next we note that \( \frac{d}{dt}(c^Tv^*(t)) = c^Ta^*(t) \). Upon substituting for \( a^* \) from equation (3.19) and further simplifying we get
\[
\frac{d}{dt}(c^Tv^*(t)) = \|v^*(t)\|^{-1}Mc^Tv^*(t).
\]
(3.20)
It can be seen from (3.20) that $c^T v^*(t)$ does not change sign. The solution to equation (3.20) is given by

$$c^T v^*(t) = c^T v_1 \ e^{\int_0^t \|v^*(\tau)\|^{-1} M \ d\tau}. \quad (3.21)$$

We now proceed to find $c$ and hence $R$ by using the transversality condition on $\lambda_r^*(t)$. As discussed in subsection (2.1), terminal conditions (3.2) and (3.3) hold with equality if the terminal velocity of the vehicle is zero and with inequality if the terminal velocity has a positive component along $e_1$.

First consider the case when the optimal trajectory terminates in rest. In this case, the terminal adjoint vector $\lambda_r^*(t_f)$ is of the form (3.10) and has a nonnegative component along $e_1$. The optimal acceleration time history thus satisfies $a^*(t_f)^T e_1 \geq 0$. Since $c^T v^*(t_f) = 0$, (3.21) implies that $c^T v_1 = 0$. Thus the only way to bring the vehicle to rest while maintaining a constant angle bisector between the acceleration and velocity vectors is to have the angle bisector perpendicular to the initial velocity vector so that the acceleration vector, which is obtained by reflecting the velocity vector about the fixed direction of angle bisector, remains opposite to the initial velocity vector for all $t \in [0, t_f]$. Thus the optimal acceleration time history takes a constant value $a^*(t) \equiv a^*$ given by

$$a^* = -M \|v_i\|^{-1} v_i. \quad (3.22)$$

Since we have shown that $a^* e_1 \geq 0$ it follows that $v_i^T e_1 \leq 0$. We therefore conclude that the optimal trajectory terminates in rest only if $v_i^T e_1 \leq 0$, that is, the required heading change is obtuse. Thus, for an obtuse angle change in heading, the minimum time and the minimum distance problems have the same solution, that is, the optimal trajectory describes a straight line and terminates in rest.

Next, consider the case when the optimal trajectory does not terminate in rest. Equation (3.11) is then satisfied only if $\alpha = 0$. This reduces (3.10) to $\lambda_r^*(t_f) = \beta e_2$, that is, the terminal adjoint vector $\lambda_r^*(t_f)$ is colinear with $e_2$ and hence perpendicular to the final velocity. It follows from (3.12) that the acceleration vector is also perpendicular to the velocity vector at the terminal time. This implies that $c$ makes an angle of $\pm \frac{\pi}{4}$ with $v^*(t_f)$. We have already shown that $c^T v^*(t)$ does not change sign. Thus we see that the angle between $c$ and $e_1$ is less than $\frac{\pi}{2}$. Next note that

$$\frac{d}{dt}(c^T J v^*(t)) = c^T J a^*(t). \quad (3.23)$$

Upon substituting for $a^*$ from (3.19) in (3.23) and further simplifying we get

$$\frac{d}{dt}(c^T J v^*(t)) = -\|v^*(t)\|^{-1} M (c^T J v^*(t)). \quad (3.24)$$

It can be seen from (3.24) that $c^T J v^*(t)$ does not change sign. Also note that $v_i^T Jc \neq 0$ for the case when the trajectory does not terminate in rest. Thus the vector $v^*(t)$ lies in the quadrant bounded by the orthogonal vectors $c$ and $\text{sign}(v_i^T Jc) Jc$ for all $t \in [0, t_f]$. Since we already know that $c$ makes an angle of $\pm \frac{\pi}{4}$ with $v^*(t_f)$, it means that the angle between $v_i$ and $v^*(t_f)$ has a magnitude less than $\frac{\pi}{4}$. So, we can conclude that the optimal acceleration time history takes the form (3.19) only when the required heading change is less than $\frac{\pi}{4}$.

Since $\frac{d}{dt}(\lambda_r^*(t)^T v^*(t)) = \|v^*(t)\| > 0$, it follows from (3.12) that the velocity component along the acceleration vector always increases. However, this velocity component is zero at the
terminal time. Hence \( a^*(t)^T v^*(t) < 0 \) for all \( t \in [0,t_f] \), that is, the angle between \( a^*(t) \) and \( v^*(t) \) is obtuse for all \( t \in [0,t_f] \). Since we have already shown that the angle between \( v_i \) and \( e_1 \) is less than \( \frac{\pi}{4} \), it follows that \( a^*(t_f) \) is opposite to the component of \( v_i \) perpendicular to \( e_1 \) and is given as

\[
a^*(t_f) = -M \left[ \|v_i\|^2 - (v_i^T e_1)^2 \right]^{-\frac{1}{2}} (v_i - (v_i^T e_1)e_1). \tag{3.25}
\]

The vector \( c \) is the unit vector along the angle bisector between \( a^*(t_f) \) and \( e_1 \), and can be obtained by substituting (3.25) in the expression,

\[
c = \|M^{-1}a^*(t_f) + e_1\|^{-1}(M^{-1}a^*(t_f) + e_1). \tag{3.26}
\]

The optimal strategy for the problem of steering the velocity vector to a desired terminal heading along the shortest path can thus be written as

\[
a^*(t) = -M\|v_i\|^{-1}v_i, \quad v_i^T e_1 \leq 0, \quad \frac{1}{\sqrt{2}} < \frac{v_i^T e_1}{\|v_i\|} < 1. \tag{3.27}
\]

where \( R = 2cc^T - I \) with \( c \) given by (3.26). Figure (3.1) shows length-optimal paths with the initial conditions \( x_i = [2,3]^T \) and \( v_i = [10,0]^T \) for different terminal headings.

![Figure 3.1: Length-optimal paths for various terminal headings](image)

4 **NUMERICAL EXAMPLE**

Consider the problem of driving the equations (2.1) and (2.2) from the initial position \( x_i = [2,3]^T \), initial speed 10 m/sec and an initial heading of 0 degrees to a terminal heading subjected to the constraint \( \|a(t)\| \leq 5 \text{ m/sec}^2 \).

Figure 4.1 shows the time-optimal and length optimal paths for a heading change of 44 degrees, while Figure 4.2 shows the speed variation along the two paths. For comparison the optimal trajectory of a vehicle moving with a constant speed of 10 m/sec to achieve a heading change of 44 degrees is also shown in Figure 4.1. Figure 4.2 shows that the vehicle moves slower along the length optimal path. For values of the terminal heading in the range 0 to 44 degrees, the numerical values of the terminal time and terminal arc length for the time-optimal and length-optimal trajectories were obtained. Figures 4.3 and 4.4 compare the terminal times and the terminal arc lengths for the time-optimal and length-optimal problems respectively, for
terminal headings upto 44 degrees. As expected, the terminal time for the time-optimal case was less than that of the length-optimal case while the terminal arc length for the length-optimal case was less than that of the time-optimal case. For the minimum time problem, the vehicle attained a heading change of 44 degrees in 1.4 seconds by covering a distance of 11.5 metres while in the minimum path length problem, the vehicle attained the heading change of 44 degrees in 1.7 seconds by covering a distance of 10 metres. In comparison, a heading change of 44 degrees performed at a constant speed of 10 m/sec under a normal acceleration constraint of 5 m/s² requires 1.54 seconds and involves covering a distance of 15.4 metres.

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6 CONCLUSION

We have considered the problem of finding time-optimal and length-optimal trajectories for a vehicle that moves in a two-dimensional plane from a given initial position and velocity to a specified terminal condition on position and/or heading subject to a magnitude constraint on the acceleration. Unlike previous works, we have considered variations in the magnitude as well as direction of the velocity vector. We have shown that the time-optimal paths for the various terminal conditions considered are either straight lines or parabolic arcs and the length-optimal paths for achieving a desired terminal heading possess a special property whereby the unit vector along the angle bisector between the velocity and acceleration vectors is a constant. We used this property to obtain a feedback strategy for the length-optimal acceleration vector that uses the instantaneous heading of the vehicle to calculate the acceleration.

References


Figure 4.1: Time and length optimal paths for a heading change of 44 degrees

Figure 4.2: Variation of velocity magnitude with time
Figure 4.3: Variation of terminal time with heading angle

Figure 4.4: Variation of terminal arc length with heading angle


